EIGENVALUE EXTENSIONS OF BOHR'S INEQUALITY

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Abstract. We present a weak majorization inequality and apply it to prove eigenvalue and unitarily invariant norm extensions of a version of the Bohr's inequality due to Vasić and Kečkić.

1. Introduction and Preliminaries

Let $\mathcal{M}_n$ denote the C^*-algebra of $n \times n$ complex matrices and let $\mathcal{H}_n$ be the set of all Hermitian matrices in $\mathcal{M}_n$. We denote by $\mathcal{H}_n(J)$ the set of all Hermitian matrices in $\mathcal{M}_n$ whose spectra are contained in an interval $J \subseteq \mathbb{R}$. By I_n we denote the identity matrix of $\mathcal{M}_n$. For matrices $A, B \in \mathcal{H}_n$ the order relation $A \leq B$ means that $\langle Ax, x\rangle \leq \langle Bx, x\rangle$ for all $x \in \mathbb{C}^n$. In particular, if $0 \leq A$, then A is called positive semidefinite.

For $A \in \mathcal{H}_n$, we shall always denote by $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ the eigenvalues of A arranged in the decreasing order with their multiplicities counted. By $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$, we denote the eigenvalues of $|A| = (A^*A)^{1/2}$, i.e., the singular values of A. A norm $|||\cdot|||$ on $\mathcal{M}_n$ is said to be unitarily invariant if $|||UAV||| = |||A|||$ for all $A \in \mathcal{M}_n$ and all unitary matrices $U, V \in \mathcal{M}_n$. The Ky Fan norms, defined as $\|A\|_{(k)} = \sum_{j=1}^{k} s_j(A)$ for $k = 1, 2, \cdots, n$, provide a significant family of unitarily invariant norms. The Ky Fan dominance theorem states that $\|A\|_{(k)} \leq \|B\|_{(k)}$ $(1 \leq k \leq n)$ if and only if $|||A||| \leq |||B|||$ for all unitarily invariant norms $|||\cdot|||$. For more information on unitarily invariant norms the reader is referred to [3].

The classical Bohr's inequality [4] states that for any $z, w \in \mathbb{C}$ and for $p, q > 1$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$,

$$|z+w|^2 \leq p|z|^2 + q|w|^2$$

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with equality if and only if $w = (p-1)z$. Several operator generalizations of the Bohr inequality have been obtained by some authors (see [1, 5, 6, 8, 11, 14, 15]). In [13], Vasić and Kečkić gave an interesting generalization of the inequality of the following form

$$\left|\sum_{j=1}^{m} z_j\right|^r \leq \left(\sum_{j=1}^{m} p_j^{\frac{1}{1-r}}\right)^{r-1} \sum_{j=1}^{m} p_j |z_j|^r, \tag{1.1}$$

where $z_j \in \mathbb{C}$, $p_j > 0$, $r > 1$. See also [10] for an operator extension of this inequality.

In this paper, we aim to give a weak majorization inequality and apply it to prove eigenvalue and unitarily invariant norm extensions of (1.1).

2. Generalization of Bohr's inequality

In this section we shall prove a matrix analogue of the inequality (1.1). We begin with the definition of the positive linear map.

A $*$-subspace of $\mathcal{M}_n$ containing I_n is called an operator system. A map $\Phi : \mathcal{S} \subseteq \mathcal{M}_n \to \mathcal{T} \subseteq \mathcal{M}_m$ between two operator systems is called positive if $\Phi(A) \geq 0$ whenever $A \geq 0$, and is called unital if $\Phi(I_n) = I_m$. Let $[A_{ij}]_k$, $A_{ij} \in \mathcal{M}_n$, $1 \leq i, j \leq k$, denote the $k \times k$ block matrix. Then each map Φ from $\mathcal{S}$ to $\mathcal{T}$ induces a map Φ_k from $\mathcal{M}_k(\mathcal{S})$ to $\mathcal{M}_m(\mathcal{T})$ defined by $\Phi_k([A_{ij}]_k) = [\Phi(A_{ij})]_k$. We say that Φ is completely positive if the maps Φ_k are positive for all $k = 1, 2, \cdots$.

To prove our main result we need Lemma 2.4 which is of independent interest. To achieve it, we, in turn, need some known lemmas.

Lemma 2.1. [12, Theorem 4] *Every unital positive linear map on a commutative C^*-algebra is completely positive.*

Lemma 2.2. [12, Theorem 1] *Let Φ be a unital completely positive linear map from a C^*-subalgebra $\mathcal{A}$ of $\mathcal{M}_n$ into $\mathcal{M}_m$. Then there exist a Hilbert space $\mathcal{K}$, an isometry $V : \mathbb{C}^m \to \mathcal{K}$ and a unital $*$-homomorphism π from $\mathcal{A}$ into the C^*-algebra $B(\mathcal{K})$ of all bounded linear operators such that $\Phi(A) = V^*\pi(A)V$.*

Lemma 2.3. *Let $A \in \mathcal{H}_n(J)$ and let f be a convex function on J, $0 \in J$, $f(0) \leq 0$. Then for every vector $x \in \mathbb{C}^n$, with $\|x\| \leq 1$,*

$$f(\langle Ax, x\rangle) \leq \langle f(A)x, x\rangle.$$

Proof. If $x = 0$ then result is trivial. Let us assume that $x \neq 0$. A well-known result [7, Theorem 1.2] states that if f is a convex function on an interval J and

$A \in \mathcal{H}_n(J)$, then $f(\langle Ay, y\rangle) \le \langle f(A)y, y\rangle$ for all unit vectors y. For $\|x\| \le 1$, set $y = x/\|x\|$. Then

$$\begin{aligned} f(\langle Ax, x\rangle) &= f\left(\|x\|^2\langle Ay, y\rangle + (1-\|x\|^2)0\right) \\ &\le \|x\|^2 f(\langle Ay, y\rangle) + (1-\|x\|^2)f(0) && \text{(by the convexity of } f\text{)} \\ &\le \|x\|^2\langle f(A)y, y\rangle + (1-\|x\|^2)f(0) && \text{(by [7, Theorem 1.2])} \\ &\le \langle f(A)x, x\rangle\ . && \text{(by } f(0) \le 0\text{)} \end{aligned}$$

□

Now we are ready to prove the following result.

Lemma 2.4. *Let $A \in \mathcal{H}_n(J)$ and let f be a convex function defined on J, $0 \in J$, $f(0) \le 0$. Then for every vector $x \in \mathbb{C}^m$ with $\|x\| \le 1$ and every positive linear map Φ from $\mathcal{M}_n$ to $\mathcal{M}_m$ with $0 < \Phi(I_n) \le I_m$,*

$$f(\langle \Phi(A)x, x\rangle) \le \langle \Phi(f(A))x, x\rangle.$$

Proof. Let $\mathcal{A}$ be the unital commutative C^*-algebra generated by A and I_n. Let $\Psi(X) = \Phi(I_n)^{-\frac{1}{2}}\Phi(X)\Phi(I_n)^{-\frac{1}{2}}$, $X \in \mathcal{A}$. Then Ψ is a unital positive linear map from $\mathcal{A}$ to $\mathcal{M}_m$. Therefore by Lemma 2.1, Ψ is completely positive. It follows from Lemma 2.2 that there exist a Hilbert space $\mathcal{K}$, an isometry $V : \mathbb{C}^m \to \mathcal{K}$ and a unital $*$-homomorphism $\pi : \mathcal{A} \to B(\mathcal{K})$ such that $\Psi(A) = V^*\pi(A)V$. Since π is a representation, it commutes with f. For any vector $x \in \mathbb{C}^m$ with $\|x\| \le 1$, $\|V\Phi(I_n)^{1/2}x\| \le 1$. We have

$$\begin{aligned} f(\langle \Phi(A)x, x\rangle) &= f(\langle \Phi(I_n)^{1/2}\Psi(A)\Phi(I_n)^{1/2}x, x\rangle) \\ &= f(\langle \Phi(I_n)^{1/2}V^*\pi(A)V\Phi(I_n)^{1/2}x, x\rangle) \\ &= f(\langle \pi(A)V\Phi(I_n)^{1/2}x, V\Phi(I_n)^{1/2}x\rangle) \\ &\le \langle f(\pi(A))V\Phi(I_n)^{1/2}x, V\Phi(I_n)^{1/2}x\rangle && \text{(by Lemma 2.3)} \\ &= \langle \pi(f(A))V\Phi(I_n)^{1/2}x, V\Phi(I_n)^{1/2}x\rangle \\ &= \langle \Phi(I_n)^{1/2}V^*\pi(f(A))V\Phi(I_n)^{1/2}x, x\rangle \\ &= \langle \Phi(f(A))x, x\rangle\ . \end{aligned}$$

□

Remark 2.5. We can remove the condition $0 \in J$ in Lemma 2.4 and assume that $\|x\| = 1$, if we assume that Φ is unital. To observe this, one may follow the same argument as in the proof of Lemma 2.4 and use [7, Theorem 1.2].

Lemma 2.6. [3, Pg. 67] *Let $A \in \mathcal{H}_n$. Then*

$$\sum_{j=1}^{k} \lambda_j(A) = \max \sum_{j=1}^{k} \langle Ax_j, x_j \rangle \qquad (1 \le k \le n),$$

where the maximum is taken over all choices of orthonormal vectors $x_1, x_2, \cdots, x_k$.

Theorem 2.7. *Let f be a convex function on J, $0 \in J$, $f(0) \le 0$ and $A \in \mathcal{H}_n(J)$. Then*

$$\sum_{j=1}^{k} \lambda_j \left(f \left(\sum_{i=1}^{\ell} \alpha_i \Phi_i(A) \right) \right) \le \sum_{j=1}^{k} \lambda_j \left(\sum_{i=1}^{\ell} \alpha_i \Phi_i(f(A)) \right) \qquad (1 \le k \le m)$$

for positive linear maps Φ_i, $i = 1, 2, \cdots, \ell$ from $\mathcal{M}_n$ to $\mathcal{M}_m$ such that $0 \le \sum_{i=1}^{\ell} \alpha_i \Phi_i(I_n) \le I_m$, $\alpha_i \ge 0$.

Proof. Let $\lambda_1, \lambda_2, \cdots, \lambda_m$ be the eigenvalues of $\sum_{i=1}^{\ell} \alpha_i \Phi_i(A)$ with $u_1, u_2, \cdots, u_m$ as an orthonormal system of corresponding eigenvectors arranged such that $f(\lambda_1) \ge f(\lambda_2) \ge \cdots \ge f(\lambda_m)$. We have

$$\begin{aligned}
\sum_{j=1}^{k} \lambda_j \left(f \left(\sum_{i=1}^{\ell} \alpha_i \Phi_i(A) \right) \right) &= \sum_{j=1}^{k} f \left(\left\langle \left(\sum_{i=1}^{\ell} \alpha_i \Phi_i(A) \right) u_j, u_j \right\rangle \right) \\
&\le \sum_{j=1}^{k} \left\langle \left(\sum_{i=1}^{\ell} \alpha_i \Phi_i(f(A)) \right) u_j, u_j \right\rangle \qquad \text{(by Lemma 2.4)} \\
&\le \sum_{j=1}^{k} \lambda_j \left(\sum_{i=1}^{\ell} \alpha_i \Phi_i(f(A)) \right) \qquad \text{(by Lemma 2.6)}
\end{aligned}$$

for $1 \le k \le m$. □

The following result is a generalization of [9, Theorem 1].

Corollary 2.8. *Let $A_1, \cdots, A_\ell \in \mathcal{H}_n$ and $X_1, \cdots, X_\ell \in \mathcal{M}_n$ such that*

$$\sum_{i=1}^{\ell} \alpha_i X_i^* X_i \le I_n,$$

where $\alpha_i > 0$ and let f be a convex function on $\mathbb{R}$, $f(0) \le 0$ and $f(uv) \le f(u)f(v)$ for all $u, v \in \mathbb{R}$. Then

$$\sum_{j=1}^{k} \lambda_j \left(f \left(\sum_{i=1}^{\ell} X_i^* A_i X_i \right) \right) \le \sum_{j=1}^{k} \lambda_j \left(\sum_{i=1}^{\ell} \alpha_i f(\alpha_i^{-1}) X_i^* f(A_i) X_i \right) \tag{2.1}$$

for $1 \le k \le n$.

Proof. To prove inequality (2.1), if necessary, by replacing X_i by $X_i+\epsilon I_n$, we can assume that the X_i's are invertible.

Let $A \in \mathcal{M}_{\ell n}$ be partitioned as $\begin{pmatrix} A_{11} & \cdots & A_{1\ell} \\ \vdots & & \vdots \\ A_{\ell 1} & \cdots & A_{\ell\ell} \end{pmatrix}$, $A_{ij} \in \mathcal{M}_n$, $1 \le i,j \le \ell$, as an $\ell \times \ell$ block matrix. Consider the linear maps $\Phi_i : \mathcal{M}_{\ell n} \longrightarrow \mathcal{M}_n, i = 1, \cdots, \ell$, defined by $\Phi_i(A) = X_i^* A_{ii} X_i,\ i = 1, \cdots, \ell$. Then Φ_i's are positive linear maps from $\mathcal{M}_{\ell n}$ to $\mathcal{M}_n$ such that

$$0 \le \sum_{i=1}^{\ell} \alpha_i \Phi_i(I_{\ell n}) = \sum_{i=1}^{\ell} \alpha_i X_i^* X_i \le I_n \,.$$

Using Theorem 2.7 for the diagonal matrix $A = \mathrm{diag}(A_{11}, \cdots, A_{\ell\ell})$, we have

$$\sum_{j=1}^{k} \lambda_j \left(f\left(\sum_{i=1}^{\ell} \alpha_i X_i^* A_{ii} X_i \right)\right) \le \sum_{j=1}^{k} \lambda_j \left(\sum_{i=1}^{\ell} \alpha_i X_i^* f(A_{ii}) X_i \right) \qquad (1 \le k \le n).$$

Replacing A_{ii} by $\alpha_i^{-1} A_i$ in the above inequality, we get

$$\sum_{j=1}^{k} \lambda_j \left(f\left(\sum_{i=1}^{\ell} X_i^* A_i X_i \right)\right) \le \sum_{j=1}^{k} \lambda_j \left(\sum_{i=1}^{\ell} \alpha_i f(\alpha_i^{-1}) X_i^* f(A_i) X_i \right) \qquad (1 \le k \le n)\,,$$

since by an easy application of the functional calculus $f(\alpha_i^{-1} A_i) \le f(\alpha_i^{-1}) f(A_i)$. $\square$

Now we obtain the following eigenvalue generalization of inequality (1.1) as promised in the introduction.

Theorem 2.9. *Let $A_1, \cdots, A_\ell \in \mathcal{H}_n$ and $X_1, \cdots, X_\ell \in \mathcal{M}_n$ be such that*

$$\sum_{i=1}^{\ell} p_i^{1/1-r} X_i^* X_i \le \sum_{i=1}^{\ell} p_i^{1/(1-r)} I_n,$$

where $p_1, \cdots, p_\ell > 0, r > 1$. Then

$$\sum_{j=1}^{k} \lambda_j \left(\left| \sum_{i=1}^{\ell} X_i^* A_i X_i \right|^r \right) \le \left(\sum_{i=1}^{\ell} p_i^{\frac{1}{1-r}} \right)^{r-1} \sum_{j=1}^{k} \lambda_j \left(\sum_{i=1}^{\ell} p_i X_i^* |A_i|^r X_i \right)$$

for $1 \le k \le n$.

Proof. Apply Corollary 2.8 to the function $f(t) = |t|^r$ and $\alpha_i = \dfrac{p_i^{1/1-r}}{\sum_{i=1}^{\ell} p_i^{1/(1-r)}}$. $\square$

Corollary 2.10. *Let $A_1, \cdots, A_\ell \in \mathcal{H}_n$. Then*

$$\left|\left|\left| \quad \left|\sum_{i=1}^{\ell} A_i\right|^r \quad \right|\right|\right| \leq \left|\left|\left| \sum_{i=1}^{\ell} p_i^{-1}|A_i|^r \right|\right|\right| \tag{2.2}$$

for $1 < r \leq 2$, $0 < p_1, \cdots, p_\ell \leq 1$ with $\sum_{i=1}^{\ell} p_i = 1$.

Proof. Taking $X_i = I_n$, $1 \leq i \leq \ell$ in Theorem 2.9 and using that $\left(\sum_{i=1}^{\ell} p_i^{\frac{1}{r-1}}\right)^{r-1} \leq \sum_{i=1}^{\ell} p_i = 1$, we have

$$\sum_{j=1}^{k} \lambda_j \left(\left|\sum_{i=1}^{\ell} A_i\right|^r\right) \leq \sum_{j=1}^{k} \lambda_j \left(\sum_{i=1}^{\ell} p_i^{-1}|A_i|^r\right) \qquad (1 \leq k \leq n). \tag{2.3}$$

Now from (2.3) and the Ky Fan Dominance Theorem, it follows that

$$\left|\left|\left| \quad \left|\sum_{i=1}^{\ell} A_i\right|^r \quad \right|\right|\right| \leq \left|\left|\left| \sum_{i=1}^{\ell} p_i^{-1}|A_i|^r \right|\right|\right|. \qquad \square$$

Next we show that the inequality (2.2) can be improved when $A, B \in \mathcal{M}_n$ in the case when $r \geq 2$.

Lemma 2.11 ([2]). *Let f be an increasing convex function on J. Then*

$$\lambda_j \left(f\left(\sum_{i=1}^{\ell} p_i A_i\right)\right) \leq \lambda_j \left(\sum_{i=1}^{\ell} p_i f(A_i)\right) \qquad (1 \leq j \leq n)$$

for all $A_1, \cdots, A_\ell \in \mathcal{H}_n(J)$ and $0 \leq p_1, \cdots, p_\ell \leq 1$ such that $\sum_{i=1}^{\ell} p_i = 1$.

Proposition 2.12. *Let $A_1, \cdots, A_\ell \in \mathcal{M}_n$ and $r \geq 2$. Then*

$$\lambda_j \left(\left|\sum_{i=1}^{\ell} A_i\right|^r\right) \leq \lambda_j \left(\sum_{i=1}^{\ell} p_i^{1-r}\,|A_i|^r\right) \qquad (1 \leq j \leq n) \tag{2.4}$$

for all $0 < p_1, \cdots, p_\ell \leq 1$ such that $\sum_{i=1}^{\ell} p_i = 1$.

Proof. Clearly

$$\sum_{i,j=1}^{\ell} p_i p_j \left(A_i - A_j\right)^* \left(A_i - A_j\right) \geq 0. \tag{2.5}$$

It follows by a direct calculation that inequality

$$\left|\sum_{j=1}^{\ell} p_j A_j\right|^2 \le \sum_{j=1}^{\ell} p_j |A_j|^2 \tag{2.6}$$

is equivalent to (2.5). Therefore (2.6) holds. Due to the function $f(t) = t^{r/2}$ is an increasing convex function, we have

$$\begin{aligned}
\lambda_j\left(\left|\sum_{i=1}^{\ell} p_i A_i\right|^r\right) &= \lambda_j^{r/2}\left(\left|\sum_{i=1}^{\ell} p_i A_i\right|^2\right) \\
&\le \lambda_j^{r/2}\left(\sum_{i=1}^{\ell} p_i \left|A_i\right|^2\right) \\
&\qquad \text{(by Weyl's monotonicity principal [3, P. 63] applied to (2.6))} \\
&= \lambda_j\left(\left(\sum_{i=1}^{\ell} p_i \left|A_i\right|^2\right)^{r/2}\right) \\
&\le \lambda_j\left(\sum_{i=1}^{\ell} p_i \left|A_i\right|^r\right) \qquad \text{(by Lemma 2.11)}
\end{aligned}$$

for $1 \le j \le n$. Now, we replace A_i by A_i/p_i to get (2.4). □

Remark 2.13. Corollary 2.10 and Proposition 2.12 are generalizations of [14, Theorem 7].

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